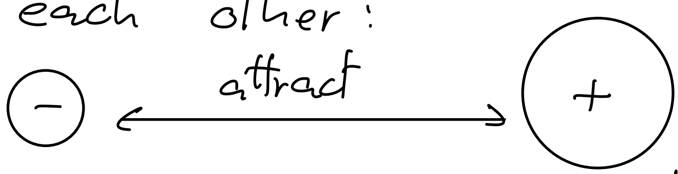
§8. Electrodynamics

Gravity is not the only "force" in the universe. In fact, we believe there are 4 fundamental forces of nature:

1) Gravity
2) Electromagnetic force
3) The strong force
4) The weak force

If you consider two electrons close to each other, they repell each other:

Similarly, two protons repell each other However, an electron and a proton will attract each other:



This force is 10^{40} times stronger than gravity!

How is it then that many protons are stuck to each other (and to neutrons) inside an atomic nucleus?

Because the "strong force" acting only along very short distances ($\sim 10^{-15}$ m) attracts them to each other (by a factor of 100 compared to electric repulsion)

There is also a "weak force" responsible for radioctivity

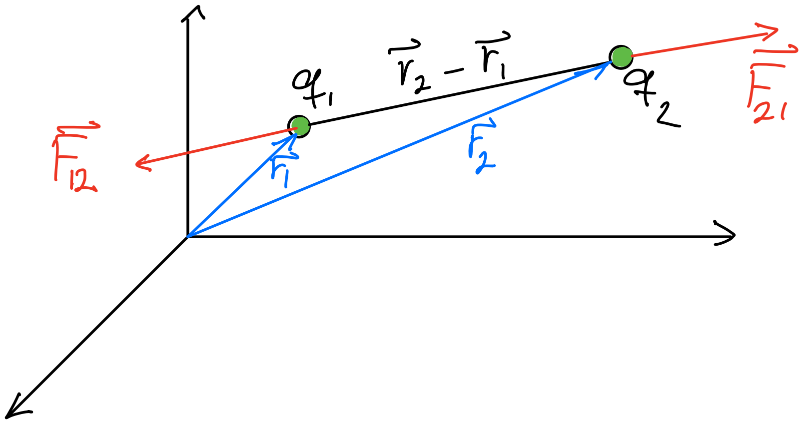
§ 8.1 Coulomb's law

How do we measure the electric charge q? What is the force between two static charges q_1 and q_2 as a function of their positions?

$\longrightarrow$ answers are contained in "Coulomb's law" (Charles-Augustine de Coulomb (1736 - 1806))

$$\vec{F}_{12} = -\vec{F}_{21} = \frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}^2} \vec{e}_{12}$$

where $r_{12} = |\vec{r}_2 - \vec{r}_1|$ and $\vec{e}_{12} = \dfrac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$



Unit of charge : 1 Coulomb [C]

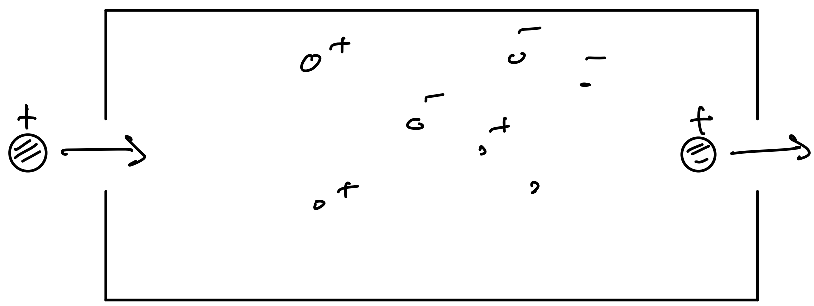
$$\frac{1}{4\pi \varepsilon_0} = 9 \cdot 10^9 \frac{N \cdot m^2}{C^2}$$

If two charges, 1 coulomb each, are separated by 1 meter, the repulsive force between them will be $9 \cdot 10^9$ N.
(weight of about 10,000,000 adults)

In these units the charge of the proton, denoted by e, is $1.6 \cdot 10^{-19}$ C and that of the electron is $-e$.

Properties of charge:

- charge is conserved "locally":
  suppose charge would be conserved non-locally:



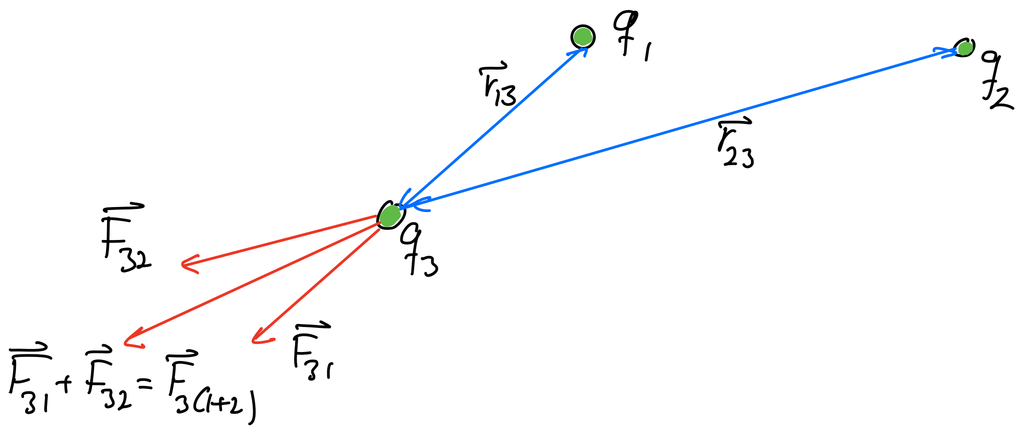
→ due to special relativity, only simultaneous in one reference frame (can choose a reference frame where there are at an instance two particles or no particle)

→ conservation laws that hold in all frames must be local

- charge is "quantized":
  All charges we have ever seen are integral multiples of a basic unit of charge, $e = 1.6 \cdot 10^{-19}\,C$

Superposition principle

Consider 3 charges q_1, q_2, and q_3 at $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$:



What will be the force on q_3 due to the other two ?

<u>answer</u> : $\vec{F}_{3(1+2)} = \vec{F}_{31} + \vec{F}_{32}$

"superposition principle"
(encountered already in Newtonian mechanics)

<u>Question</u>: Suppose we want to verify Coulomb's law. How will you confirm the dependence of the force on q_1, q_2, and $r = |\vec{r}_1 - \vec{r}_2|$?

Think about it.

The ratio of gravitational to electric forces

Claim: F_g/F_e, ratio of gravitational to electric forces, $\sim 10^{-40}$

$\uparrow$

How do we obtain this answer?

Consider two particles of masses m_1 and m_2 and charges q_1 and q_2, a distance r apart.

$$\longrightarrow \frac{F_g}{F_e} = \left(\frac{G m_1 m_2}{r^2}\right) \cdot \left(\frac{q_1 q_2}{4\pi \varepsilon_0 r^2}\right)^{-1} = \frac{G m_1 m_2 \cdot 4\pi\varepsilon_0}{q_1 q_2}$$

For two electrons, this gives:

$$\frac{F_g}{F_e} = \frac{6.7 \cdot 10^{-11} \cdot (9 \cdot 10^{-31})^2}{(1.6 \cdot 10^{-19})^2 \cdot 9 \cdot 10^9} \simeq 2.3 \cdot 10^{-43}$$

proton and electron: $\frac{F_g}{F_e} \sim 10^{-40}$

For two protons: $\sim 10^{-36}$

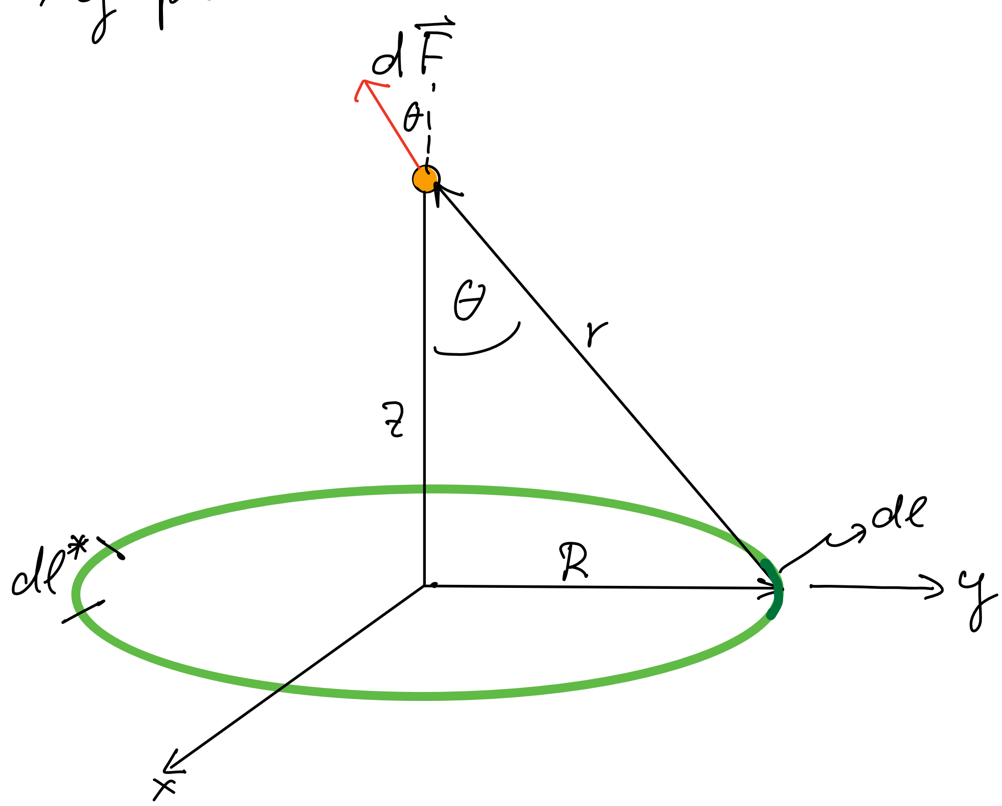
But: in macroscopic bodies, electric charges come in pairs and cancel!

That's why the earth and the moon attract each other by gravity!

Coulomb's law for continuous charge density

Example 1:

Consider a circular wire of radius R with λ coulombs per meter, lying in xy-plane :



Find the electric force it exerts on a point charge q located at height z on the z-axis.

Solution:

$$|d\vec{F}| = \frac{q \cdot \lambda \, dl}{4\pi \varepsilon_0 (R^2 + z^2)}$$

Only need keep the vertical part of the force, the horizontal part is cancelled by dl^*:

$$F_z = \int dF_z = \int \frac{q \cdot \lambda \, dl}{4\pi \varepsilon_0 (R^2 + z^2)} \times \frac{z}{\sqrt{R^2 + z^2}}$$

$$= \frac{q \cdot \lambda \cdot 2\pi R z}{4\pi \varepsilon_0 (R^2 + z^2)^{3/2}}$$

where the factor $\frac{z}{\sqrt{R^2+z^2}} = \cos\theta$ projects out the vertical part of dF and $2\pi R$ is the integral over dl.

Tests:

1) set $z = 0 \longrightarrow$ should give zero by symmetry and indeed it does!

2) for $z \gg R$ the wire should appear like a point charge, it does!

§8.2 The electric field $\vec{E}$

Let us write the force q_2 due to q_1:

$$\vec{F}_{21} = \frac{q_1 q_2}{4\pi \varepsilon_0 r_{12}^2} \vec{e}_{12}$$

$$= \frac{q_1}{4\pi \varepsilon_0 r_{12}^2} \vec{e}_{12} \cdot q_2 = \vec{E}(\vec{r}_2)\, q_2$$

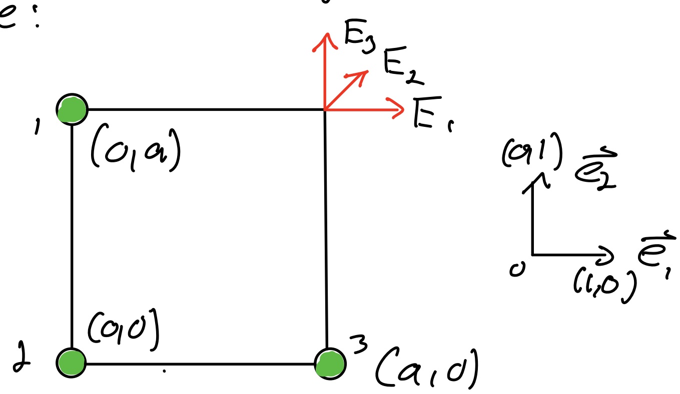
$\vec{E}(\vec{r}_2)$: "the electric field at the location of q_2"

Step 1: The charge q_1 produces a field $\vec{E}(\vec{r}_2) = \frac{q_1}{4\pi \varepsilon_0 r_{12}^2} \vec{e}_{12}$ at the location of q_2

Step 2: The charge q_2 responds to the field by feeling a force $\vec{F}_{21} = q_2 \vec{E}(\vec{r}_2)$

Example 2:

compute $\vec{E}$ with charges q at $(0,0)$, $(a,0)$, $(0,a)$ of a square:



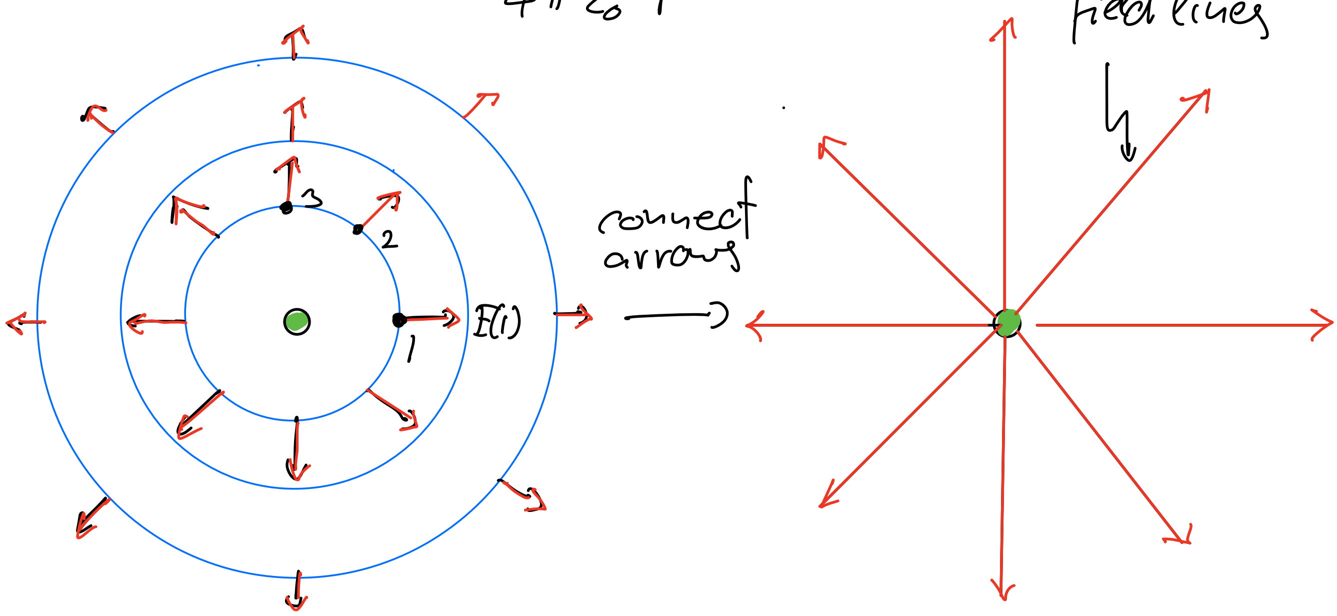
where we have:

$$\left.\begin{aligned}
\vec{E}_1 &= \frac{q}{4\pi\varepsilon_0 a^2}\,\vec{e}_1 \\
\vec{E}_2 &= \frac{q}{4\pi\varepsilon_0 2a^2}\,\frac{\vec{e}_1+\vec{e}_2}{\sqrt{2}} \\
\vec{E}_3 &= \frac{q}{4\pi\varepsilon_0 a^2}\,\vec{e}_2
\end{aligned}\right\} \Rightarrow \quad \vec{E}(a,a) = \frac{q}{4\pi\varepsilon_0 a^2}\left(1+\frac{1}{2\sqrt{2}}\right)(\vec{e}_1+\vec{e}_2)$$

Visualizing the field

For a point charge at the origin, the electric field is:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\,\vec{e}_r = \frac{q}{4\pi\varepsilon_0 r^3}\,\vec{r}$$

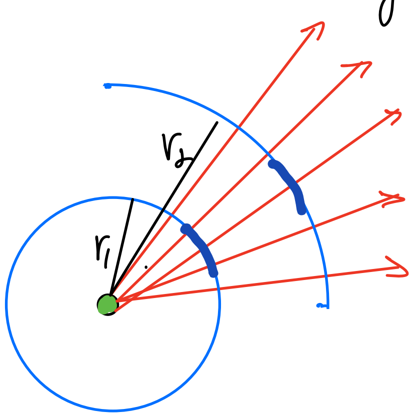


Question: Do we lose information about the electric field at a poit $\vec{r}$ if we connect arrows to lines?

1) We retain the information about the "direction" of the electric field, i.e. the direction of the force vector acting on a "probe charge"

2) It seems that we are losing information about the "magnitude" of the force

But: "density" of field lines through a fixed area ΔA contains information about magnitude!



area of a sphere in $\mathbb{R}^3$ grows as r^2 with the radius r

$\to$ density of N field lines at distance r:

$$\sim \frac{N}{r^2} \sim |\vec{E}|$$

Proposition 1:

The electric field of a point-charge is a conservative field. That is, the force on a probe charge is a conservative force.

Proof:

According to Theorem 3 in §4, a force is conservative if it can be written as $\vec{F} = \vec{\nabla} U(\vec{r}) = \begin{pmatrix} \partial_x U \\ \partial_y U \\ \partial_z U \end{pmatrix}$

$\rightarrow$ for the electric field $\vec{E}$ we have to show: $\vec{E} = -\vec{\nabla} V$

Claim: $V(r) = \dfrac{q}{4\pi \varepsilon_0 r}$

We compute:

$$-\frac{\partial V}{\partial x} = -\frac{q}{4\pi\varepsilon_0} \frac{\partial(1/r)}{\partial x} = -\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right]$$

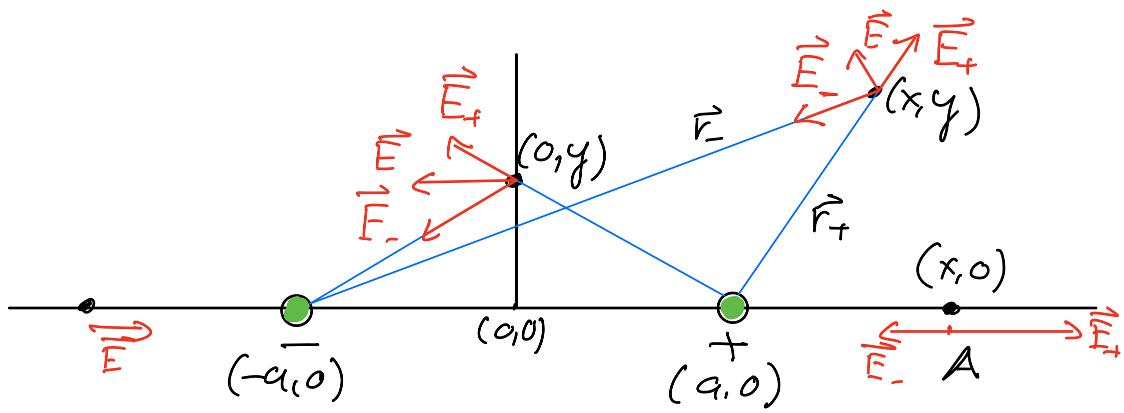
$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{2}\right]\frac{2x}{(x^2+y^2+z^2)^{3/2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{x}{r}$$

$$\Longrightarrow -\vec{\nabla} V = \frac{q}{4\pi\varepsilon_0 r^2} \underbrace{\frac{\vec{e}_x\, x + \vec{e}_y\, y + \vec{e}_z\, z}{r}}_{=\vec{e}_r} = \vec{E}(\vec{r})$$

$\square$

Example 3 (field of dipole):

Consider a charge q at $(a,0)$ and a charge $-q$ at $(-a,0)$. Consider the field at a generic point (x,y).



From the picture, we see:

$$\vec{E}_+ = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_+}{r_+^3} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{e}_1(x-a)+\vec{e}_2 y}{((x-a)^2+y^2)^{3/2}}$$

$$\vec{E}_- = -\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}_-}{r_-^3} = -\frac{q}{4\pi\varepsilon_0}\frac{\vec{e}_1(x+a)+\vec{e}_2 y}{((x+a)^2+y^2)^{3/2}}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec{e}_1(x-a)+\vec{e}_2 y}{((x-a)^2+y^2)^{3/2}} - \frac{\vec{e}_1(x+a)+\vec{e}_2 y}{((x+a)^2+y^2)^{3/2}}\right]$$

For a point A with $x>a$, we obtain:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\vec{e}_1\left(\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\,\vec{e}_1\,\frac{4ax}{(x^2-a^2)^2}$$

$$=: \frac{\vec{p}}{4\pi\varepsilon_0}\,\frac{2x}{(x^2-a^2)^2} \quad \text{where } \vec{p} = 2aq\,\vec{e}_1$$

"dipole moment"

$\longrightarrow$ for $x \gg a$, we get:

$$\vec{E}(x \longrightarrow \infty) \simeq \frac{\vec{p}}{2\pi\varepsilon_0 x^3} = \frac{\vec{p}}{2\pi\varepsilon_0 r^3}$$